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A discrete classical space–time could require 6 extra-dimensions

Philippe Guillemant *, Marc Medale, Cherifa Abid

Aix-Marseille Université, CNRS, IUSTI UMR 7343, 5 rue Enrico Fermi, 13453 Marseille Cedex 13, France

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ABSTRACT

We consider a discrete space–time in which conservation laws are computed in such a way that the density of information is kept bounded. We use a 2D billiard as a toy model to compute the uncertainty propagation in ball positions after every shock and the corresponding loss of phase information. Our main result is the computation of a critical time step above which billiard calculations are no longer deterministic, meaning that a multiverse of distinct billiard histories begins to appear, caused by the lack of information. Then, we highlight unexpected properties of this critical time step and the subsequent exponential evolution of the number of histories with time, to observe that after certain duration all billiard states could become possible final states, independent of initial conditions. We conclude that if our space–time is really a discrete one, one would need to introduce extra-dimensions in order to provide supplementary constraints that specify which history should be played.

0. Introduction

One of the fundamental problems in physics is the uncertainty propagation inherent to non-linear dynamical systems, which leads to their unpredictability. Certainly due to the complexity of the n-body problem even with point-like masses, according to our best knowledge no numerical simulation has been published to compute explicitly the uncertainty propagation during time. This problem is generally apprehended using mathematical approaches such as Gaussian Mixture Models and Polynomial Chaos Expansions [1]. We propose here a numerical model that enables to compute the

* Corresponding author.
E-mail address: philippe@guillemant.net (P. Guillemant).

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uncertainty propagation of the n-body problem by considering the non point-like case of interactions into a billiard.

It is widely considered that unpredictability is a limitation to calculability that does not question determinism, but which is only due to our ignorance of adequate precision of initial conditions (supposed to be infinite in a continuous space). This calculability problem is apprehended by different approaches, essentially stochastic ones that deal with probabilities of multiple possible evolutions or histories. These probable histories are not considered to be realistic solutions but only approximations or estimations of the unique solution that really occurs. This raises thus the following question: *could the information about initial conditions for objects in our physical reality be accurate enough to admit systematically unique solutions?* As long as one avoids this question, one automatically neglects the possibility for the evolution of a system to be really potentially multiple, meaning that this multiplicity would be inherent to physical reality. Although this is questioning determinism, it is no more possible today to ignore this, because the Everett multiverse theory [2], claiming that all possible events due to indeterminism occur into many worlds, is now widely considered as one of the best interpretation of quantum mechanics [3,4].

Neither questioning nor cautioning the validity of this interpretation, one can anyway consider the potential incompleteness of discrete time-dependent laws of physics that it implies. This potential incompleteness is going hand in hand with a lack or a loss of information that could determine the actual history, face to which random choices seem to be made like in Brownian motion. A remarkable result mathematically demonstrated recently [5] is that the deterministic equations of elastic hard-spheres shocks do not prevent from a true Brownian motion to be reached after a long time into a billiard. This demonstration has been made by studying the branching process of collision trees following an infinitesimal perturbation of the position of a tagged ball. These authors have shown that this phenomenon occur when the number of balls is increased toward infinity [5]. From a physical point of view the occurrence of a Brownian motion corresponds to a loss of "memory" of any phase state, meaning it becomes a random state independent of initial conditions. But what is this "memory" or initial condition, except information? What is the physical nature of information? Could information be really physical?

A confusing problem about *physical information* is that it is closely linked to entropy and energy in a way that leads to contradictions when one simply tries to define it. First, the idea that information is physical has been proposed by Szilard [6] and highlighted by Landauer [7,8], giving information the thermodynamic sense of a physical quantity, which is the opposite of entropy variation. It is based on the fact that erasing one bit would correspond to the dissipation of $k\ log(2)$ in entropy. But the Shannon's theory of information [9] has introduced subjective entropy [10] that is defined using probabilities, quantifying all types of information such as that contained in a message. Consequently, the attempts to define physical information as a genuine physical quantity, avoiding the trap of probabilities that inevitably represents a subjective lack of knowledge [11], have resulted in a lot of confusion, amplified by the eternal debate about how to solve the Maxwell's Demon problematic [12,13]. Today, a widespread opinion is to consider physical information as related to the computational complexity of a system [14], expressing it for example as the entropy of a cellular automaton that leads to its calculation [15]. But this interesting concept of physical information, which seems to have the advantage of being more objective, is lacking a computational model of universe. Furthermore, the physicality of information as originally defined by Szilard has been verified experimentally in 2012 [16] and confirmed by new experiments more recently [17]. However, today the debate about the subjectivity of entropy is far from being closed [18].

Indeed, if phase information is actually physical, its density should be everywhere limited as it is the case for energy or matter: objects with infinite energy or matter do not exist. But this seems to contradict our idea of a continuous space, which could be indefinitely divided into elementary volumes. According to Beckenstein [19] and Penrose [14], for example, our space–time could be a discrete universe that has a finite density of information everywhere.

In this paper, we use a finite density of information thanks to a discrete space model where position and momentum phase states define the physical information of the system. Then, one can replace subjective microstate probabilities of the system (denoted p_i) by objective ones that are calculated from phase-state uncertainties. Following Brilloin [20], this means considering entropy as the amount of missing information. If all microstates are equiprobable one can convert Gibbs entropy into its Boltzmann expression and then to information entropy [21], as follows:

$$S = -k \sum_{i} p_i \ln(p_i) = k \ln(W) = k \ln(2)(I_{max} - I_{pq}) \tag{1}$$

In this expression I_{pq} is the physical information of the system, varying from a maximum value $I_{max} = Log_2(Wmax)$ corresponding to the situation where all objects of the system are perfectly configured (no uncertainty), to the zero minimum corresponding to the situation where uncertainties on phase states are maximum.

The aim of this paper is to study the possibility for a truly multiple evolution of trajectories in our discrete space to be a consequence of the decay of I_{pq} below a critical value. To enlighten this assertion, we present computations of isolated billiards, which essentially consists in determining the critical time step separating deterministic and stochastic trajectories. An idealized billiard has been chosen to study the influence of key parameters on the loss of information as it is among the simplest toy models related to basic physical laws. The main result of this study is a quantification of the amount of deterministic information compared to the amount of information contained into initial conditions. This is followed by an evaluation of the growing law of the number of multiple histories after critical step, so as to estimate the number of extra dimensions that could be necessary to maintain determinism.

1. Theoretical framework

1.1. Billiard model and governing equations

We have chosen a frictionless 2D hard-sphere billiard model with elastic shocks (same mass and diameter) to estimate the propagation of uncertainties during interactions. We have worked with two kinds of billiards, a periodic one and a finite size one with borders (cf. Fig. A.1 in appendix). Between two shocks the equations of movement of any ball i involve phase position $\overrightarrow{q_\imath(t)}$ and velocity $\overrightarrow{V_\imath(t)}$ according to the following equations:

$$\left|\overrightarrow{V_\imath(t)}\right| = Cte \quad \text{and} \quad \overrightarrow{q_\imath(t_1)} = \overrightarrow{q_\imath(t_0)} + \int_{t_0}^{t_1} \overrightarrow{V_\imath(t)} dt \tag{2}$$

If we consider any collision between two balls i and j with $\overrightarrow{p_\imath}$ and $\overrightarrow{p_\jmath}$ momentums at collision point P_c (defined as the middle of their centers $G_\imath$ and G_j, cf. Fig. 1), they are changed to $\overrightarrow{p'_\imath}$ and $\overrightarrow{p'_j}$. The momentum conservation equation gives $\overrightarrow{p_\imath} + \overrightarrow{p_\jmath} = \overrightarrow{p'_\imath} + \overrightarrow{p'_\jmath}$, i.e. two scalar equations in projection along x and y axis. A third equation is given by conservation of kinetic energy (same ball masses) $p_1'^2 + p_2'^2 = p_1^2 + p_2^2$. Two others equations are given by the conservation of momentum in projection along the $\overrightarrow{\Delta_2}$ axis (perpendicular to $\overrightarrow{\Delta_1} = \frac{\overrightarrow{G_1G_2}}{\left|\overrightarrow{G_1G_2}\right|}$ illustrated in Fig. 1): $\overrightarrow{p'_\imath} \cdot \overrightarrow{\Delta_2} = \overrightarrow{p_\imath} \cdot \overrightarrow{\Delta_2}$ and $\overrightarrow{p'_j} \cdot \overrightarrow{\Delta_2} = \overrightarrow{p_\jmath} \cdot \overrightarrow{\Delta_2}$, but only one of them is required, as the other is not independent of previous ones.

1.2. Uncertainty propagation

To calculate the propagation of uncertainties on momentum $\overrightarrow{p_\imath}$ and position $\overrightarrow{q_\imath}$, we compute them from the phase differences between two quasi-identical billiards (hereafter denoted $\Omega 1$ and $\Omega 2$). The first billiard is set up with random initial position and momentum of every ball. The second one is set up as the first one, but with perturbated ball positions obtained by adding initial position uncertainties +/-ε (plus or minus are also randomly chosen) where the quanta ε determines the information density.

The momentum uncertainties illustrated in Fig. 2 are calculated at the collision times, corresponding to discontinuous variations of momentum uncertainties, from $\Delta p = \left|\overrightarrow{p_{\imath 2}} - \overrightarrow{p_{\imath 1}}\right|$ before a shock to $\Delta p' = \left|\overrightarrow{p_{\imath 2}'} - \overrightarrow{p_{\imath 1}'}\right|$ after the shock. Note that the amplification $\Delta p' > \Delta p$ is due to distinct collision

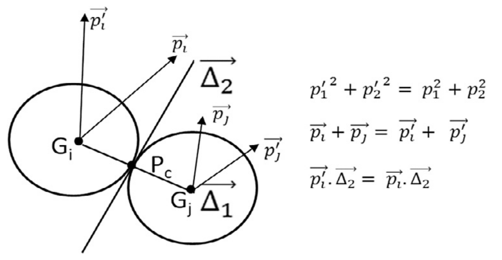

Fig. 1. Sketch of a contact between two balls.

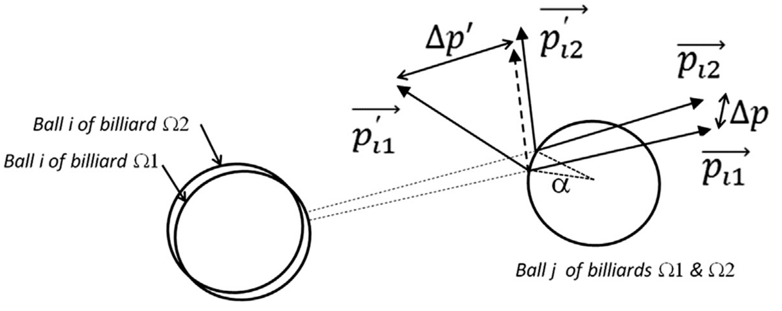

Fig. 2. Propagation of momentum uncertainties Δp to $\Delta p'$ during a collision.

points that form an angle α between ball j center (identical for $\Omega 1$ and $\Omega 2$ on Fig. 2). They remain constant between two shocks, unlike position uncertainties $\overrightarrow{\Delta q_{\imath 1}}$ and $\overrightarrow{\Delta q_{i2}}$ that are growing linearly between two shocks.

2. Computations

2.1. Discretization and phase information

The set of governing equations is computed in a discrete Cartesian phase-space with the 64 bits precision of our computer. To study the influence of information density on the amount of information we set a range of different ε quanta of position (from 2^{-5} to 2^{-35}), equal to the minimum position uncertainty at the beginning of calculations. Our calculations were done into a 1024×1024 square-pixels billiard game of 10 bits resolution. The precision of initial conditions varies from 15 bits to 45 bits with a 5 bits increment.

The aim of our calculations is to measure the uncertainty evolution versus time by computing them from the differences from billiard Ω_1 and Ω_2 ball phases. Given $\Delta p(n)$ and $\Delta q(n)$ uncertainties of ball n, the amount of phase information I_{pq} can be expressed in the form:

$$I_{pq} = \sum_{n=1}^{N_b} Log_2\left(\frac{\Delta pmax}{\Delta p\,(n)}\right) + \sum_{n=1}^{N_b} Log_2\left(\frac{\Delta qmax}{\Delta q(n)}\right) \tag{3}$$

where $\Delta pmax$ and $\Delta qmax$ are the maximum uncertainties: $\Delta qmax = \Delta pmax = 1024$. Note that we always have: $\Delta pmax \geq \Delta p(n) \geq \varepsilon$ and $\Delta qmax \geq \Delta q(n) \geq \varepsilon$.

As uncertainties on positions create uncertainties on momentum and conversely the two summations should be comparable. Our calculations confirmed that one can consider them as approximately equal. As individual uncertainties and their logarithms are very different when n varies, we calculated their geometrical average $\Delta pmoy$, whose logarithm is the average of $Log(\Delta p(n))$ for all balls, leading to the following expression of I_{pq}:

$$I_{pq} = 2N_b\, Log_2(\frac{\Delta pmax}{\Delta pmoy}) \tag{4}$$

2.2. Algorithm

To calculate the time variation of I_{pq}, we have computed all successive shocks between ball i_1 and ball j_1 in billiard $\Omega 1$ and repeated these calculations in the second billiard $\Omega 2$ until one obtains $i_2 \neq i_1$ or $j_2 \neq j_1$, meaning that we have reached a critical step where histories in the two billiards become different, as in Fig. 3 which displays divergent positions and velocities of blue and red balls of the two billiards.

We performed these calculations for different ball numbers N_b ranging from 2^3 to 2^9 and different values of ε, ranging from 2^{-5} to 2^{-35}. Then, for each couple (N_b, ε) we repeated the computations N_r times with random editions of initial conditions ($N_r = 4096/N_b$). This choice was a compromise between the necessity to reduce computational times while getting statistically robust enough results. Illustratively, our calculations last approximately a few days on the Windows XP system based computer we used. The algorithm for calculations until critical step consists in:

– Calculate the next shock n in billiard $\Omega 1$ by computing all the possible shocks of the ball i_1 with other balls of $\Omega 1$ at different times, choose the minimum time t_1 and then get the value of j_1;

– Calculate the corresponding shock into billiard $\Omega 2$ and then get its time t_2;

– If $t_1 < t_2$ calculate the position of balls i_2 and j_2 at time t_1, or else calculate the position of balls i_1 and j_1 at time t_2, so as to get position uncertainties $\Delta q(n)$ at the same minimum time and to compute the change in balls positions after time $|t_2 - t_1|$.

– Compute the momentum uncertainties $\Delta p(n)$ from new momentums in $\Omega 1$ and $\Omega 2$ after times t_1 and t_2 of collisions.

– Add the logarithm of $\Delta p(n)$ so as to calculate their geometrical average $\Delta pmoy\,(N_t)$:

$$Log(\Delta pmoy\,(N_t)) = \frac{1}{N_r N_b} \sum_{m=1}^{m=N_r} \sum_{n=N_t N_b}^{n=N_t(N_b+1)} Log(\Delta p\,(n)) \tag{5}$$

N_t is the non-dimensional time corresponding to the average number of shocks per ball. N_t varies from 0 to N_c where N_c is the critical time, i.e. the average number of shocks per ball at critical step. N_r is the number of random editions of initial conditions.

In the following we present the main results from our study: (i) the evolution of phase information I_{pq} until critical step N_c; (ii) the variation of N_c with respect of main parameters ε and N_b; (iii) the paradox of information at critical step. We also present in appendixes more detailed results: different approaches to extrapolate computed results when N_b tends toward infinity, along with the influence of void ratio R_v (defined as the ratio between balls area to billiard area). Its role in the loss of information becomes very important when R_v tends to zero as it particularly accelerates the loss rate. In our study we have chosen a high value of $R_v = 0.33$ corresponding to minimum rates of information loss. We made this choice in order to minimize computing times.

3. Critical step

3.1. Evolution of phase information

Despite the fact that $\Delta p(n)$ values vary considerably for one ball, we found that the evolution of phase information until critical step can be statistically considered as linear. This can be explained by the fact that while uncertainties stay very low compared to the ball diameter, the dispersion angle α in Fig. 2 remains close to zero and then ball curvature has still no effect on the shock dispersion law of uncertainties, whose averages are growing geometrically with the same rate.

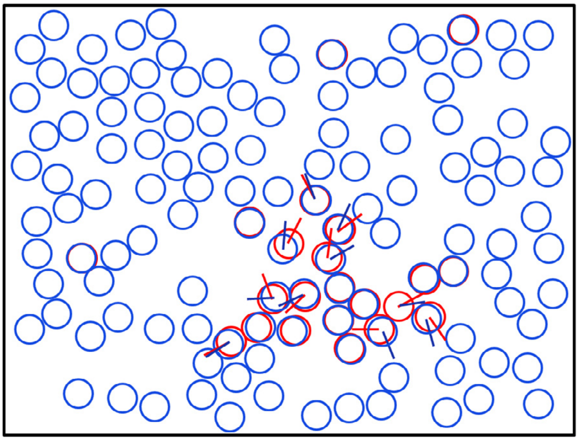

Fig. 3. Illustration of the critical step with $N_b = 128$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

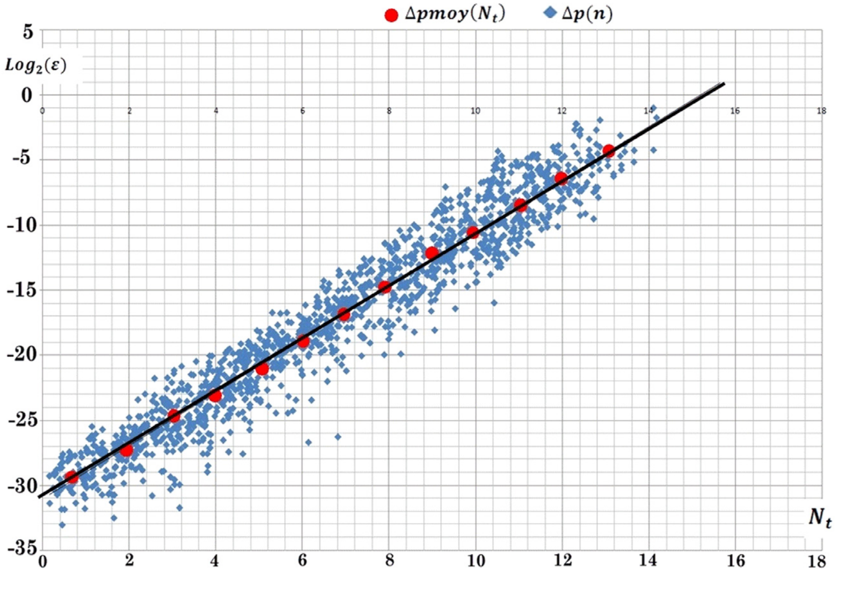

Fig. 4. Example of $\Delta p(n)$ samples and their average values $\Delta pmoy\,(N_t)$ computed during one calculation cycle until critical step, with $N_b = 128$ balls and $\varepsilon = 2^{-35}$. Critical step reached at $N_c = 14$ shocks per ball. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In Fig. 4 the linearity of $\Delta pmoy$ (red dots) can be observed with $\Delta p(n)$ values (blue dots) that were computed for $\varepsilon = 2^{-35}$ and $N_b = 128$. For visibility of blue dots we used here only one random edition of initial conditions ($N_r = 1$). With a high number N_r of these editions all results show this linear evolution, even with low values of N_b. As a consequence, computing this linear variation of I_{pq} versus time is equivalent to study the influence of main parameters on the critical step N_c.

If one splits the precision of results, defined on the one hand by the required precision ($P_c = 10$ bits) for deterministic trajectories and on the other hand by additional precision ($P_a = -Log(\varepsilon)$ bits) using the maximum of information density for initial coordinates (varying from 5 to 35 bits), one can synthesize this in a relationship for phase information I_{pq}, using the time N_c at critical step:

$$I_{pq} = P_c + P_a(1 - \frac{N_t}{N_c}) \tag{6}$$

After critical step, the distance separating two coupled balls suddenly increases and the two billiard balls are no longer superposed on the screen (Fig. 3). At this stage, our calculations are stopped because it is no longer possible to synchronize the two billiard balls so as to calculate the uncertainties at well defined times. However, it is obvious that after this step, phase information rapidly decays to zero because velocity vectors of diverging $\Omega 1$ and $\Omega 2$ coupled balls become extremely different and cause a contagion effect that rapidly spreads to the rest of billiard. Thus we can consider that phase information quickly reaches its zero minimum and then adopt the linear expression of I_{pq} (10) as a rough approximation when $N_t > N_c$.

3.2. Evolution of critical step versus main parameters

The main parameters we consider here are $Log_2(N_b)$ and $Log_2(\varepsilon)$, whose effects on the critical step are opposite and summarized in Fig. 5. We made three types of calculation to obtain these results, which are differentiated in this figure by rectangles:

- A real computation of all shocks between all balls in the billiard, giving results until $N_b = 512$, detailed in Appendix A;
- A modeled simulation using a momentum dispersion function, giving results until around $N_b = 130\,000$, detailed in Appendix B;
- An asymptotic linear approximation for higher values of N_b, detailed in Appendix C.

We note a good correspondence between the results from the different simulations, except for high values of N_b and low values of ε for which critical time N_c is the highest. These discordances for $N_c > 10$ can be explained by the limited computing precision of 64-bit, which introduces arbitrary information to lower bits in our calculations at each shock. The consequence of such a computing error, after its propagation to higher level bits for numerous shocks, is that our modeled simulation becomes more reliable than our computations, because only the latter is affected by the bias of calculating all shocks.

If we consider restrictive domain values of parameters we then notice that the variation of N_c with $Log_2(N_b)$ and $Log_2(\varepsilon)$ is approximately linear, as confirmed in appendix (C.1). For example, if we suppose that $Log_2(N_b) > 10$ and $Log_2(\varepsilon) < 50$, we can write:

$$Nc \sim 2.8 + 0.21P_a - 0.35\, Log_2(N_b) \tag{7}$$

In our frictionless hard-sphere billiard with equal diameters and masses, the void ratio R_v is the last parameter that influences the N_c value. As Fig. B.1 (in Appendix B) illustrates, when the diameter of balls is decreased by a factor 2 (or its area by a factor 4), the average value of the amplification factor of the momentum uncertainty is approximately increased by a factor 10. This is decreasing the value of N_c or increasing the rate of information loss in the same proportions. Note that the critical time is decreased only in terms of shocks, but as the real time between shocks is increased by the rarefaction of shocks, the non-reduced critical time is increased.

3.3. Paradox of information at critical step

Knowing the critical step N_c, we can now focus on the main practical interest of our study, which is to compare the amount of information contained into initial conditions and the maximum amount of information to compute reliable trajectories of balls, here called "calculable information". We consider here that a trajectory is no more reliable when it is computed beyond the critical step, because it would be biased by the problem of emergence of multiple histories. This point is generally ignored or neglected by computations implying a lot of interactions like in gaz dynamics for example, because they always use high level statistical equations that occults the limit of deterministic calculability.

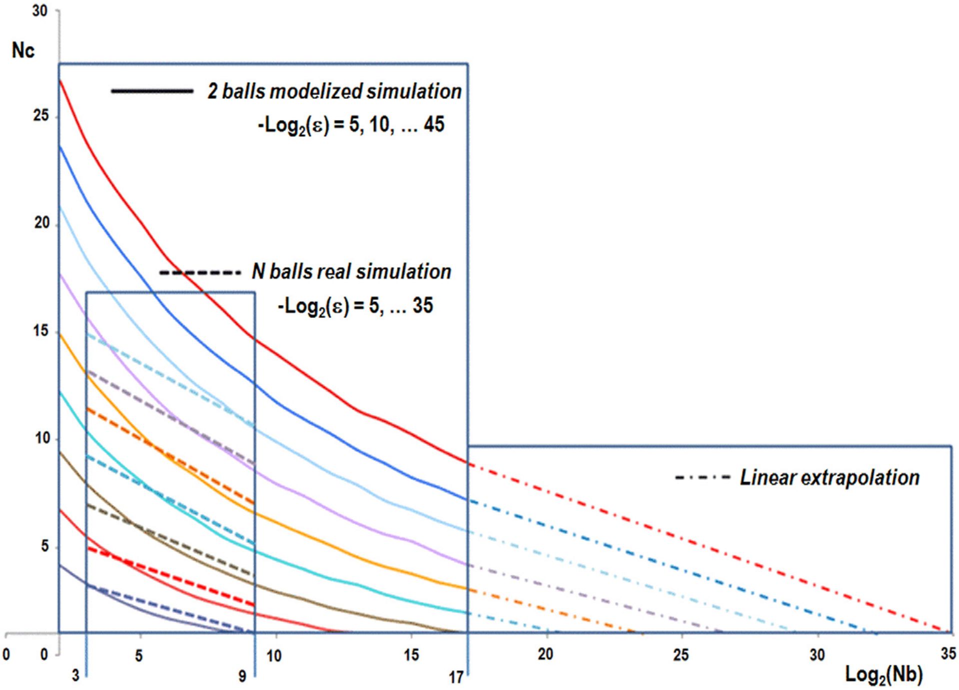

Fig. 5. N_c variation curves versus $\text{Log}_2(N_b)$ et $-\text{Log}_2(\varepsilon)$ for the void ratio $R_v = 0.33$: Lines represent a modeled simulation (Appendix B); dotted lines on the left represent the real computation of all shocks; dotted lines on the right represent an extrapolation of N_b values and the red curve on the top corresponds to $\varepsilon = 45$: see Appendices A–C for detailed results.

The amount of calculable information I_{ci} can be expressed knowing only the number of coordinates of shocks, which are enough to determine all trajectories. So, for one axis we have:

$$I_{ci} = P_c N_b N_c \tag{8}$$

The information I_{ic} contained into initial conditions must include both positions and velocities and use the total precision $P_i = P_a + P_c$ so that we have:

$$I_{ic} = 2P_i N_b \tag{9}$$

This leads us to rise what we call a "paradox of information at critical step", meaning that it exists large sets of parameters N_b, P_c and P_i for which the amount of calculable information is lower than the amount of information contained into initial conditions. This unexpected situation is occurring when $N_c < 2P_i/P_c$ ($P_i < 64$ bits in our study).

For example, if we consider an initial precision $P_c = 10$ bits for calculable information, this is occurring for $P_i = 40$ bits when the number of balls is greater than 1024 ($Log_2(N_b) = 10$). The Fig. 6 shows the higher the various precisions, the higher should be N_b for the paradox to occur. It also means that whatever the prescribed precision, as small as desired, there is always a number of balls above which the calculable information is lower than the one used to store initial conditions. Indeed, increasing N_b has the effect to increase the rate of information loss. Fig. 7 illustrates an example of this rate growing from an average of 3 bits per shock, for low values of N_b, to more than 5 bits per shock for high values of N_b, corresponding to the cases $N_b = 200$ and 5000, respectively. The intermediate case ($N_b = 1000$) corresponds to the paradox of information limit ($I_{ic} = I_{ci}$) for which the number of horizontal blocks is the double of the number of vertical blocks.

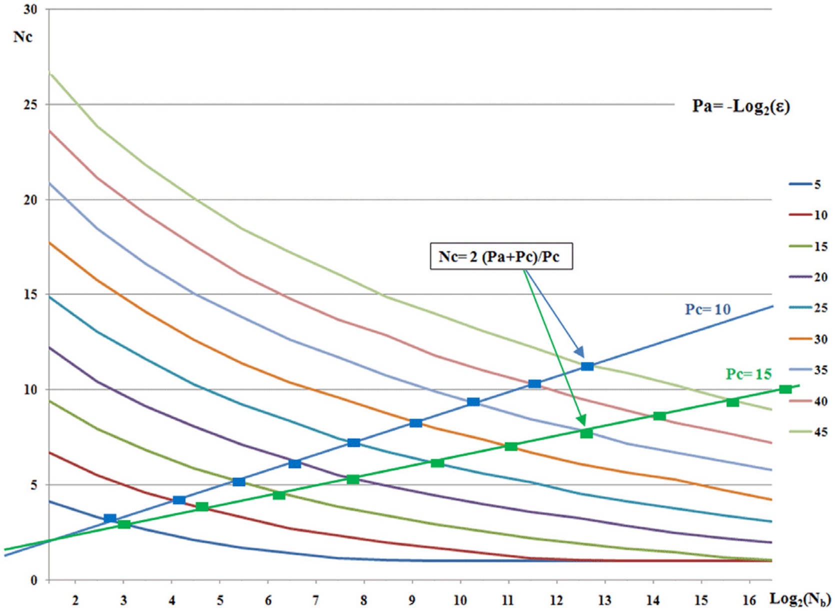

Fig. 6. Information paradox and its emergence conditions: squares correspond to N_c and N_b values for paradox limit.

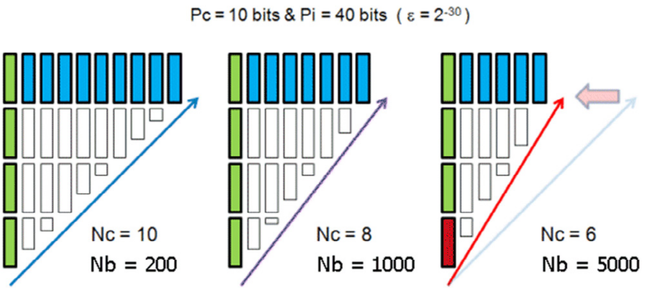

Fig. 7. Average loss of information of one phase coordinate. Colored rectangles are 10-bit blocks. Green blocks correspond to the information I_{ic} required for initial conditions ($P_i = 40$ bits). Blue blocks correspond to the calculated information I_{ci} ($P_c = 10$ bits). The number of horizontal blocks is the critical time N_c. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This paradox of information can be explained by the increased probability for any ball at any time to reach the critical step when the number of balls is increased. This is due the fact that our estimation of the critical step N_c has only a statistical sense. If one considers individual ball trajectories, the possibility to reach the critical step, even after only one shock, is never zero and only depends on initial conditions.

Our computations have essentially pointed out that the amount of deterministic or calculable information is of the same order as the amount of information that is contained into initial conditions. Our study of parameters can be summarized by two key thresholds:

- A critical step, above which multiple shock histories appear, separating deterministic history and stochastic histories;

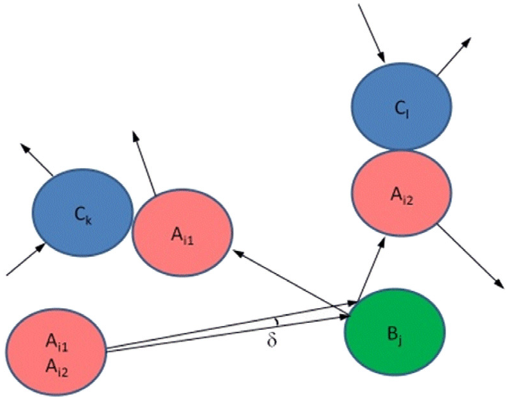

Fig. 8. First occurrence of multiple billiard histories after critical step.

- A critical number of balls, above which the calculable information of the deterministic history becomes lower than the information contained into initial conditions.

We will develop this result in the discussion as an argument in favor of the incompleteness of physical laws in a discrete space–time of finite density. Note that it is reinforced by a more qualitative result concerning the void ratio parameter R_v, which has an analog effect. When the ball diameters are decreased, R_v tends toward zero and then the rate of loss of information grows toward infinity, as Fig. B.1 illustrates in Appendix B.

3.4. Growing law of different histories after critical step

After critical step, histories of shocks in the two billiards begin to differ, exactly after the first shock between a couple of balls (A_i, B_j) is followed by subsequent shocks of A_i involving different couples (A_i, C_k) and (A_i, C_l) with k # l, as illustrated in Fig. 8.

Consequently, phase indeterminations of position and velocity of A_i become too large for going on along a unique history by synchronizing the two billiard balls, i.e. for calculating for each couple (A_i, B_j) the two slightly different times of Ai subsequent shock with a unique C_k. This is due to the large amplitude of the angle indetermination δ (cf. Fig. 8), which is rapidly converging toward its maximum 360°.

Then, we have to consider multiple histories, but it is no more possible in practice to make statistically efficient calculations because the computing time is multiplied by the number of histories and then growing up exponentially with the number of shocks. However, it remains interesting to estimate the law of growing of the number of histories versus time, by considering the neighborhood C_k of each ball A_i for which each couple (A_i, C_k) is involved as a possible subsequent shock. For doing this, let us first introduce a new stage corresponding to the Brownian diffusion time where the total amount of balls phase information in the billiard has reached zero. It means that each ball of the billiard has reached 360° indetermination and also passed, after each individual critical step, through a diffusion process by traveling a distance whose maximum is $\Delta qmax$, so that not only the velocity but the position uncertainties too have reached their maximum. After this diffusion time all the possible histories become independent of initial conditions. Therefore the information that could redefine the completely lost positions and velocities of each ball in each history become an additional one, brought by a "choice" of one history among all the possible. As a consequence, after the Brownian diffusion time a Brownian evolution is really installed in histories, in agreement with the result of Bodineau et al. [14].

Now, to estimate the growing law of the number of histories (or "choices"), let's D being the mean number of C_k balls (k varying from 1 to D) having a shock with A_i after the Brownian diffusion step.

The Brownian evolution authorizes us to assimilate D to a constant after this step. As D varies with void ratio its calculation is not straightforward, but we just need to know that $D > 1$ for any billiard with at least 3 balls, to conclude that the number Nh of histories is growing exponentially with time or number of shocks:

$$Nh \sim NhcsD^{(N-Ncs)} \tag{10}$$

where $Nhcs$ is the total number of shocks until global diffusion step and Ncs the total number of different histories at this step. It is then important to note that whatever the number of balls N_b and the quanta ε, the number of states of the billiard keeps bounded, meaning it is not growing with time.

Conjugating this fact with (i) an exponential growth of the number of histories with time, and (ii) the more and more homogeneous distribution of phases of all histories occurring during Brownian motion has an original consequence. Above a certain saturation time, the number of states reached by all histories exceeds the total number of possible states of the billiard. It means that the final state of the billiard becomes completely independent of its initial conditions and that any state of the billiard can become a final state.

4. Discussion

The main interest of the present work is to have identified and quantified, with our different expressions of N_c, a critical step above which calculations are no more deterministic, meaning that the precision of initial conditions does not allow to pursue calculations without introducing, among the multiple possibilities, an arbitrary choice of the final state, independent of initial data. We have also shown that if calculations are still extended enough in time, whatever state of the billiard could become a final state.

This raises fundamental questions in the context where information would be really physical and then its amount bounded. Though it involves a simplified discrete model, the concept of physical information that we introduced in (7) models a classical type of uncertainty that would be inherent to any discrete space–time, into which the loss of physical information would no more be a subjective loss of information but a real loss of information of space–time itself. That is why it is important to discriminate the genuine multiplicity of histories that could be due to the physicality of information (of limited density), from a stochastic unpredictability inherent to a practical limit of initial conditions precision.

A significant result of our work is that we have found that the amount of deterministic information that can be calculated until the critical step is of the same order as the amount of information that is contained into initial conditions. Then, we have raised a "paradox of information" in as much as it expresses a strange situation: the deterministic information that can be extracted from fundamental laws has a maximum that can be much lower than the entire amount of information contained into the initial conditions. Beyond the fact that the initial data should have a physical limit, any predictive model should indeed be able to provide a calculation which plays the role of data compression algorithm. In particular, it should be able to compress the data relative to trajectories of balls in a billiard into a set of initial conditions occupying much less memory, yet we observe the opposite. We think that this is seriously arguing in favor of the idea that our "known" laws of the Universe are in fact incomplete at the discrete level.

As a consequence, we have shown that dealing with physical information raises a strange situation, which is to make the final state of the billiard independent of its initial state after a saturation time. This independence is not so strange for statistical physics which made the choice to base its powerful equations upon random trajectories, justifying probabilistic calculations. But this choice is not solving the fundamental problem of indeterminism, which implies that if one wants to describe a unique evolution among a multiverse of possibilities, one has to introduce additional parameters that play the role of additional space–time dimensions. So, it raises this question: for a discrete space whose dimension is Nd, how many dimensions should be added to restore determinism?

Adding Nd dimensions so as to also include final positions of a 2D billiard balls ($Nd = 2$) seems inescapable to determine precisely trajectories followed by balls after a saturation time. But it is not enough, because we also have to choose a unique billiard history among the multiple ones which inevitably connect initial and final given states, when time is still increasing. The estimation of extra-dimensions is then complicated by the possibility of cyclic eternal returns that reproduce periodically the same final state. However, such eternal cycles do not exist in our real space–time because it is in expansion. But is it possible to generalize thus our study, knowing that the perfectly isolated billiard does not exist in nature! A first reason to do so is that the loss of information that is responsible for multiple histories of our billiard is not limited to elastic shocks and can be generalized to all types of interactions, because we just used momentum and energy conservation laws for our calculations. An isolated discrete space–time would then lack information that should be compensated by extra-dimensions. Second, note that even for a non-isolated system, the decoherence mechanism which reduces quantum superpositions into physical information issued from environment does not prevent undetermined choices.

If we make this generalization, there is a more convenient way to estimate extra-dimensions, which is to put initial conditions into the past and to define the path itself as crossing the present state of the universe (time of observation). If we suppose its uniqueness, we then need the usual $Nd + 1$ dimensions to define the path and we still have to add 2 Nd dimensions to connect it to past and future remote conditions. In practice, it could consist in specifying the coordinates of only one object, whose past and future positions are also unique into the multiverse, so as to assure a deterministic version of our entire Universe. One would then need 3 $Nd + 1$ dimensions[1] to uniquely define each history:

- Nd dimensions to define initial conditions;
- Nd dimensions to define final conditions;
- $Nd + 1$ dimensions to define the path through present state.

Note that velocity data is not needed here because the path is considered unique. In classical physics, the past and future positions of an object are considered as determined by present positions and velocities. The use of velocities replaces that of final conditions, supposing determinism and then a unique universe, without extra-dimensions. But this hypothesis does not permit to characterize remote past or future positions in a discrete space–time, because physical information is lost between the present and the past, or the present and the future.

5. Conclusion

In a discrete space–time of finite density of information, it turns out from our computations that the amount of deterministic information that is calculable using physical laws is of the same order as the amount of information that is contained into initial conditions. This suggests a possible incompleteness of governing laws at discrete level. Our results then imply that a 3D discrete space–time would need 3 additional dimensions to specify final conditions, and even 6 extra ones if one supposes the existence of alternative present paths, like in many-worlds theory. In particular, we have argued for the possibility for final conditions of a sufficiently distant future of our universe to be at least partially independent of our present state, which seems interesting if only because it would preserve a chance for free will.

Though it is attractive to characterize a unique version of our Universe within the multiverse by postulating the uniqueness of its present state, one can wonder how many extra-dimensions are necessary. Our work suggests a discrete space–time could require up to 6 extra-dimensions. This should be interpreted as highlighting the fact that our known physical laws of the Universe could still be incomplete to describe reality, and that we would need complementary laws. As it should be in the present case timeless ones, we guess that quantum gravity emerging from Wheeler Dewitt equation [22] could bring key elements to compensate the loss of information, so as to calculate extra dimensions and then restore determinism.

[1] That is 4 extra dimensions for a 2D billiard and 6 ones in realistic 3D cases. This addition of $2 \times Nd$ dimensions is necessary to add two points so as to uniquely define the state of the multiverse in present and future, or in past and future, depending on the reference frame of observation (past or present).

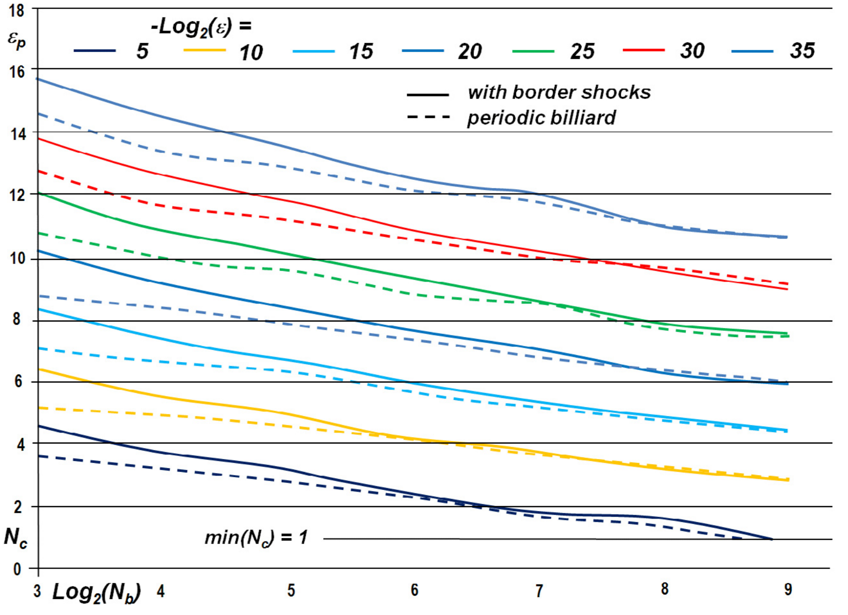

Fig. A.1. Results of statistical study using the calculation of all shocks repeated N_r times with random editions of initial coordinates. Two billiard models were used: the first one (unbroken lines) by taking into account shocks with the borders and the second one (dotted lines) using a periodic billiard. Note negligible border effects for $N_b > 128$ and approximately linear evolution of N_c with $\text{Log}_2(N_b)$ and $\text{Log}_2(\varepsilon)$.

Appendix A. Periodic and aperiodic billiards computation

Parameters: Void ratio $R_v = 0.33$, Abscissa = number of balls N_b ranging from 8 to 512. Ordinate = initial uncertainties ranging from $\varepsilon = 2^{-5}$ (bottom) to $\varepsilon = 2^{-35}$ (top). Number of random editions of initial conditions = $N_r = 4096/N_b$.

Appendix B. Modelized simulation

The calculation of critical step can be made without computing all the shocks, knowing the statistical distribution of moment uncertainty amplification during a shock. We used Fig. B.1 slope parameters ($Sd = 2.2$, $Su = -5$) of the triangular shape distribution obtained for void ratio 0.33 ($R = 16$) to calculate the evolution of N_c for higher values of N_b (until 2^{17}) and lower values of ε (until 2^{-45}), by randomizing the amplification factor with respect to the distribution, and multiplying it from $\Delta p = \varepsilon$ to $\Delta p = 1$. Smooth statistical results of Fig. 5 (second rectangle) were obtained by repeating this operation a thousand times.

$$A = Abs(128 + 4Log_2\left(\frac{\left|\overrightarrow{p_{i2}}' - \overrightarrow{p_{i1}}'\right|}{\left|\overrightarrow{p_{i2}} - \overrightarrow{p_{i1}}\right|}\right)) \tag{B.1}$$

Different distributions of A value were calculated for different void ratios R_v (from 0.33 to 0.0013) obtained by decreasing the radius R of billiard balls from 16 to 1. Ne is the number of samples corresponding to each integer in abscissa A. Note the important increase of A when R is decreasing: when R varies from 16 to 1 the average value of the distribution varies in the same way as void ratio, from about 0.1 ($R_v = 0.33$) to about 0.001 ($R_v = 0.0013$). A was calculated using a large set of random

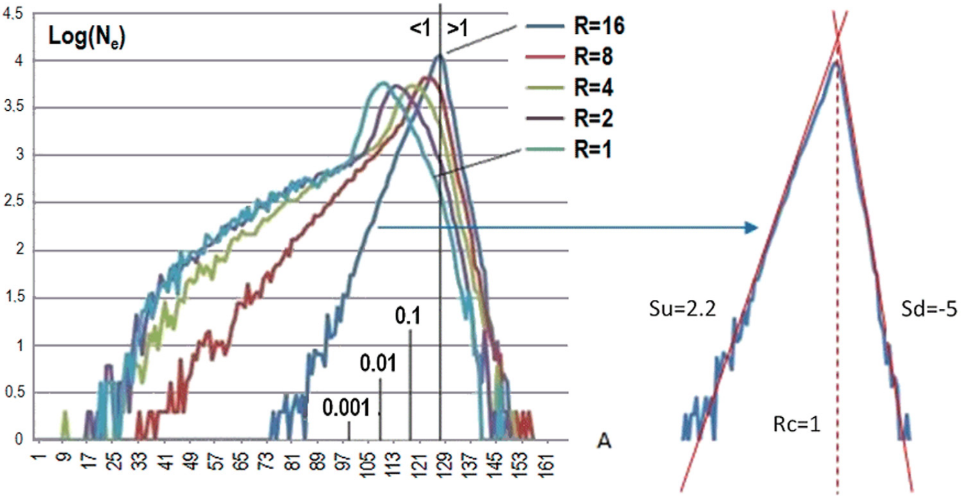

Fig. B.1. Distribution of the amplification factor A of momentum uncertainty $\left|\overrightarrow{p_{i2}} - \overrightarrow{p_{i1}}\right|$ during a shock.

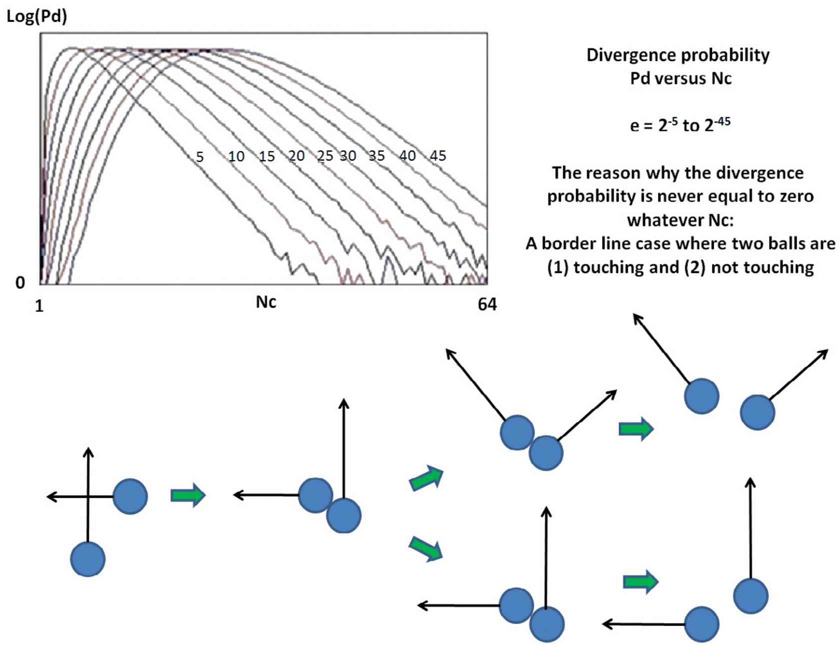

Fig. C.1. Top left, logarithm distributions of the probability P_d of divergence (critical step) after a number of shocks N_c, in function of ε. At the bottom, illustration of the borderline case where two balls are just touching, which explains why P_d is never equal to 0 whatever the N_c value.

editions until 100 000 shocks were computed. The coefficients 128 and 4 of (B.1) were chosen in order to work with integer and positive values of the abscissa A and to get a sufficient set of sampled values on the horizontal axis.

Appendix C. Asymptotic behavior

The asymptotic behavior of N_c curves when $N_b \rightarrow \infty$ is evaluated by taking into account that whatever ε, all N_c curves bisect the axis $N_c = 1$. One reason for this is that the first three curves calculated for $\varepsilon = 2^{-5}$, $\varepsilon = 2^{-10}$ and $\varepsilon = 2^{-15}$respectively bisect the axis $N_c = 1$ when $Log_2(N_b)$ is approximately equal to 9, 12.5 and 16 (Fig. 5). A second reason, illustrated by Fig. B.1, is that whatever ε it is always possible to find a value of N_b for which the probability to reach the critical step after only one shock is around 100%, so that the result is $N_c = 1$.

Fig. C.1 then show the result of a statistical computation of the probability distributions of N_c when ε varies from 2^{-5} to 2^{-45}, where all curves always bisect the vertical axis $N_c = 1$. This is justified by the existence of a borderline case in the behavior of coupled balls: a shock occurs for one of them but not for the other, as Fig. C.1 shows: this borderline case involves two balls which either collide or just graze, with velocity values becoming totally different whatever the ε *value*, which explains why the critical step can be reached for $N_c = 1$.

A linear model is then used in Fig. 5 to approximate the decrease of N_c when N_b tends to infinity:

$$N_c \sim A + B\,Log_2(\varepsilon) + C\,Log(N_b) \tag{C.1}$$

where A, B and C are approximately constant values in restrictive intervals of $Log_2(N_c)$ and $Log_2(\varepsilon)$, equal to 2.8, 0.21 and 0.35 in relation (7), approximately verified when $Log_2(N_b) > 10$ and $Log_2(\varepsilon) < 50$.

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